Ice Sheets III: Membrane Stresses, Thermomechanical Processes, and Grounding Line Instability

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Importance of Membrane Stresses

The complete force balance for glacial ice (neglecting viscoplastic effects) is the Stokes Balance:

$$\nabla \cdot \sigma - \rho g = 0 \tag{1}$$

And the Cauchy stress tensor σ (or rather, the deviatoric stress tensor $\tau = \sigma - pI$) depends on the strain rate $\dot{\varepsilon}$ through Glen's law. Solving this equation on an ice sheet scale is prohibitive, though. Historically, ice sheets have been modeled by approximating the Stokes equations with the Shallow Ice Approximation (SIA), which assumes that only vertical shear stresses are prevalent. However, this approximation falls down at the Membrane Coupling Length,

$$L = \left(\frac{H}{\tau_d}\right)^{\frac{n}{n+1}} \left(\frac{u}{A}\right)^{\frac{1}{n+1}}, \tag{2}$$

where so-called "membrane stresses" (the τ_{xx}, τ_{yy}, and τ_{xy} members of the deviatoric stress tensor) become important. Here H is the ice thickness, τ_d is driving stress, u is velocity and A is the Glen's Law constant. Membrane stresses are found to play a strong role in regulating the thermomechanical instability mechanism for ice stream formation, as discussed below.

The First Order approximation to the Stokes balance includes both vertical shearing and membrane stresses (although it does not include the higher-order stress terms τ_{zx} and τ_{zy}, which are involved in non-hydrostatic forces). This balance, while much simpler than Stokes, still requires the solution of at 3-dimensional PDE for velocities. (Meanwhile SIA does not require a PDE solution for velocities, and the Shallow-Shelf Approximation, which includes only membrane stresses, requires the solution of a 2-dimensional PDE.)

Hindmarsh (2004, JGR) details a host of approximations to First Order that do not require 3-dimensional solves, and finds that the balance L1L2, which effectively combines SIA and membrane stresses while requiring only a 2-dimensional PDE solve for velocities, approximates Stokes well down to length scales of 8-10 ice thicknesses.

Glacial Thermodynamics

The Glen's Law parameter has a strong dependence on ice temperature, and so modeling temperature evolution in ice is quite important. Ice temperature follows the heat equation:

$$\theta_t + \vec{u} \cdot \nabla\theta = \frac{1}{\rho_i c}\nabla \cdot (k\nabla\theta) + \frac{D}{\rho_i c}, \tag{3}$$

where c is the heat capacity, k is the heat diffusivity, and D is the strain heating rate, equal to one-half the trace of the tensor $\dot{\varepsilon} \cdot \tau$. Typically surface temperatures are given by atmospheric temperature, and the basal boundary condition is either Dirichlet (if at the freezing point) or Neumann (if frozen). The solution may involve a temperature boundary layer, depending on the magnitude of the surface (or basal) mass balance a. The Peclet number $\frac{Ha}{k}$, which is a measure of the ratio of vertical heat advection to diffusion, determines the width of the boundary layer.

Horizontal advection can be important, too, if there is a gradient in surface temperature due to stratification in the atmosphere. In this case relatively cold ice can be advected from upstream and give a temperature minimum in the interior.

Thermomechanical Surging and Streaming

Using a flowline $(x - z)$ thermomechanical SIA model, *Payne* (1995) showed that ice sheets can undergo surging of the type suggested in *MacAyeal* (1993). In this mechanism, an ice sheet frozen to its base builds up overtime, and its base, shielded from the cold atmosphere and heated from below by a geothermal flux, warms to the freezing point. The ensuing rapid sliding heats surrounding ice through basal frictional heating and the sliding region grows upstream, and the sheet surges and rapidly thins. Finally the surging is arrested when the bed, no longer insulated, refreezes and slowly starts growing again. This surging is thought of as a possible explanation for the Heinrich events seen in the paleo record.

However, attempts to view this mechanism in a 3-dimensional Shallow Ice model resulted in gridscale dependency, where the streams were as thin as resolution would allow. *Hindmarsh* (2004, JFM) investigated the linear stability of such a system. He found stability for large transverse length scales and short along-flow length scales. However, no shortwave cutoff was found for transverse length scales, suggesting that SIA was not sufficient, since membrane stresses would become important at these scales. Results from time-dependent 3-dimensional thermomechanical SIA model runs (see Lecture Slides) show that the number of spontaneously occuring ice streams depends on the resolution used. However, when membrane stresses are included via the L1L1 approximation from *Hindmarsh* (2004, JGR), the number of streams is independent of resolution (providing the grid spacing is below a certain length), suggesting more physical behavior.

Grounding Line Dynamics

Parameterizations of grounding line processes are important for ice sheet and ice shelf dynamics. Of particular significance are processes that advance or retreat the grounding line. Such grounding line movements are essential to understanding marine ice sheet behavior over a range of time scales, from the millenial to the decadal.

The region surrounding the grounding line is complex in many respects. Basal geometry is one such complicating factor, since specific features of the bedrock topography can influence grounding line dynamics. In addition to being mobile over a complex basal bed, advancing or retreating to a variety of external forcing factors, the grounding line is also a region of high gradients in the velocity and stress fields. Such factors require that the region surrounding the grounding line be treated as a boundary layer.

The region of the grounding line, in fact, may be treated as two distinct boundary layers. The first is a region upstream of the grounding line that may be treated using the **spreading slab ice approximation**:

$$C|u|^{\frac{1}{l}-1}u - \frac{\partial}{\partial x}\left(2HB|e_{xx}|^{\frac{1}{n}-1}\frac{\partial u}{\partial x}\right) = \rho g H \epsilon \tag{4}$$

In (4), the first term and second terms on the left-hand side represent the basal traction and the longitudinal stress, respectively. The right-hand side contains the driving stress. The boundary conditions in this region are a velocity continuity condition on the upstream side,

$$u^* = u \tag{5}$$

and a stress continuity condition on the downstream side,

$$\tau_{xx} = \frac{1}{4}\rho g\left(1 - \frac{\rho}{\rho_w}\right)H - \tau_{xx}^{BACK}. \tag{6}$$

The boundary region downstream of the grounding line is known as the **membrane stress boundary layer**. Within this region, the governing equation for the flow may be written as follows:

$$\frac{\partial F}{\partial x} = \frac{1}{4}\rho g \frac{\partial H^2}{\partial x} + T_b \tag{7}$$

Here, the term on the left-hand side is the longitudinal stress, while the terms on the right-hand side are the driving stress and basal traction, respectively. Surprisingly, the membrane stress boundary layer must be modeled with a basal traction term (even though the underside of the ice is not in touch with the bedrock) in order to prevent blow-up of the solution in finite time.

The Schoof formulation (*Schoof,* 2007, JGR) for the membrane stress boundary layer may be written as follows:

$$\frac{\partial}{\partial x}\left(2A^{-1/n}H\left|\frac{\partial u}{\partial x}\right|^{\frac{1}{n}-1}\frac{\partial u}{\partial x}\right) = \rho_i g H \frac{\partial s}{\partial x} + C|u|^{m-1}u \tag{8}$$

$$q = \left(\frac{A(\rho_i g)^{n+1}(1-\rho_i/\rho_w)^n}{4^n C}\right)^{\frac{1}{m+1}} H_G^{\frac{m+n+3}{m+1}} \tag{9}$$

Equation (8) describes the flow, and is analogous to equation (7). As in the basic formulation, the Schoof flow equation also includes a basal traction term, a necessity for constraining flow within the membrane stress boundary layer. Equation (9) represents a parameterization for the mass flux of material, q, through the membrane stress boundary layer. This formulation, though seemingly non-intuitive, is nevertheless successful at constraining the mass flux and preventing blow-up of the solution across this crucial region. Though successful for some modeling applications, the Schoof parameterization is inadequate for modeling rapid movements in the grounding line position and grounding line curvature.

The Schoof formation has been used by *Pollard & DeConto* (2009) to model the large-scale movement of the grounding line in the West Antarctic Ice Sheet (WAIS) over millennial time scales. Their numerical study, which modeled the evolution of ice over the entire Antarctic continent over 5 million years, found that the Antarctic ice sheet appeared to have three stable modes:

- **Mode 1** Thick grounded ice sheets extending to the continental slope in both the East and West Antarctic.
- **Mode 2** Complete disappearance of the WAIS, and significant retreat of the East Antarctic Ice Sheet.

- **Mode 3** An intermediate state similar to the present state of the Antarctic, with large ice shelves in the West Antarctic (Ross and Ronne-Filchner), and grounded ice extending to the continental shelf over much of the East Antarctic coast.